

Black Holes, Information and the Universal Coefficient Theorem

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This note is to bring to the reader's attention the fact that general relativity and quantum mechanics differ from each other in one main aspect. General relativity is based on the diffeomorphism covariant formulation of the laws of physics while quantum mechanics is constructed such that its fundamental laws remain invariant to a change of topology. It is the goal of this paper to show that in order to obtain a complete description of quantum gravity one has to extend the principle of diffeomorphism invariance from general relativity in the sense of quantum mechanics i.e. the laws of physics must be covariant to a change in the topology of spacetime. On the practical side, I provide an answer to the black hole information paradox: the missing information is permanently encoded in the higher cohomology of the quantum field space allowed by the given situation. Notions like entanglement become dependent on choices of coefficients in cohomology. There remains still an uncertainty principle, namely the one given by the incompatibility of various choices of coefficients required for the construction of the cohomology. In this way the answer to the unitarity problem appears to be : indeed, unitarity is restored, as required by standard quantum mechanics but one still needs an extended uncertainty principle (envisioned already by Hawking) which has an exact formulation in the universal coefficient theorem.

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INTRODUCTION

The prescriptions of general relativity and quantum mechanics are taking away most of the absoluteness associated to choices of coordinates, trajectories followed by particles and states of physical systems in the absence of any accessible information about them. It is my observation that there still remains an epistemological defect associated to these ideas. Not to all arbitrary conventions has been taken their absolute status away. In fact the connectivity of space is probably the last convention that still is considered absolute by many physicists. It is my observation that one cannot assign an absolute topology to spacetime in the absence of a method for detecting such a topology. This obstruction is at the origin of several paradoxes and inconsistencies in the formulation of quantum gravity, notably the "information paradox" for black holes. Because of this, in order to construct a consistent formulation of physics in a general context, it appears to be necessary for the laws of nature to be specified in a topology-covariant way. The attempt of doing so is the main subject of this paper. In a more practical tone, one of the problems arising in the discussion of black holes in a quantum field theoretical context is the fact that the quantum prescription of unitarity may be lost in processes involving the thermal radiation of black holes [1]. In fact it can be shown that in a semi-classical approximation, each process involving the presence of a horizon may lead to outgoing thermal radiation [2]. An in-falling pure quantum state is then mapped into the external radiation which presents a thermal spectrum thus violating unitarity. I analyze here the origin of this problem and find that the semi-classical approximation is insufficient for a correct quantum description of phenomena involving space-time horizons. In fact,

the solution appears to be related to topological properties of the transformation groups considered as acting on the given space. The covariant formulation with respect to some transformations and the related ideas leading to equivalence principles (Galilei, Lorentz, Poincare) are important in this context. In particular, it is possible to relate the existence of a simple manifest covariant formulation and, in a more extended way, of an equivalence principle, to some topological properties of the transformation groups employed in the theory. If a topological covariant formulation of a theory is required, (as I assume to be the case in the context of black holes and horizons) its existence will depend on the structure of the torsion (*Tor*) and extension (*Ext*) groups associated to a coefficient structure in cohomology. The main requirement will be for the measurable physical properties to be independent of choices of coefficients in cohomology and hence independent of the apparent topology induced by these choices. This condition will introduce a set of factors (distinct factors for distinct extensions) in the canonical quantization conditions and in the Bogolubov transformations [7]. It can be shown that these factors will change the thermal nature of the emergent radiation in a way that can appear only when analyzing its topological properties (the cohomology). The practical conclusion of this article is that the violation of unitarity is an artifact generated by the semi-classical nature of the approximations used until now. Once one takes various topological effects into account, in a manifestly topologically covariant way, the black hole radiation is corrected with non-thermal terms and an avoidance of unitarity breaking becomes possible. As a side remark, it will also be visible that the information can be seen as encoded in the cohomology of the space in a dimension smaller by one unit. This is in agreement with the

present formulation of the holographic principle and one of its realizations (the AdS/CFT correspondence [8]).

COVARIANCE PRINCIPLES IN PHYSICS

The main developments of the past century (special relativity, general relativity and quantum mechanics) have brought to our attention the fact that abstract mathematical conventions should not stand at the fundamentals of a description of reality. In general, the role of conventions is to facilitate the comprehension of physical reality and not to assign physical reality to conventional constructions. This idea was noted probably for the first time by Einstein and incorporated in his theory of special relativity as the weak equivalence principle: "the laws of nature should not depend on the arbitrary choice of an inertial reference frame". This law was further generalized to the statement that "the general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatsoever (generally co-variant)" [3]. This statement can be translated in modern terminology by using (co)homological algebraic notations. In order to do this let me follow reference [4] and define

$$\mathcal{P} = Tr_4 \circ L \quad (1)$$

to be the Poincare group where Tr_4 is the four dimensional translation group and L the Lorentz group and

$$G = Tr_4 \circ L_G \quad (2)$$

to be the Galilei group where again Tr_4 is the four dimensional translation group and L_G is the group of galilean boosts and rotations. In contrast to the Poincare group, due to the absoluteness of time, the Galilei group admits several semi-direct structures. One can use for example the decomposition

$$G = (((Tr_3 \otimes B_3) \circ T)) \circ \mathcal{R} = H \circ \mathcal{R} \quad (3)$$

where Tr_3 is the 3 dimensional translation group, B_3 is the 3 dimensional boost group, T represents time translations and \mathcal{R} represents rotations. This allows one to define the mechanical evolution space as the homogeneous space parametrized by (t, x, \dot{x}) . This evolution space is however not a homogeneous space for the Poincare group, because of the different cohomological properties of the Galilei and Poincare groups: while $H_0^2(G, U(1)) = \mathbb{R}$ for the Galilei group, for the Poincare groups $H_0^2(\mathcal{P}, U(1)) = 0$. This difference in the cohomological structures of the Galilei and Poincare groups has as consequence the absence of any simple 'covariant' formulation of Newtonian mechanics, as opposed to the Poincare case [4]. In this way, the existence of a special topological structure of the symmetry group of a theory is

related to the existence of a simple enough covariant formulation. This is not to say that a covariant formulation for the Newtonian mechanics is impossible. In fact, it is possible, after certain choices regarding the probing of topological properties are made. It is important to notice how this argument can be extended when one deals not only with invariance with respect to a symmetry group but with invariance to a change in the measurement technique for the topology of space-time. I showed previously [9] that the observed cohomological structure depends on choices of arbitrary coefficient groups. The difference in the algebraic prescriptions induced by different choices of coefficient groups is generally encoded in universal coefficient theorems (UCT). These theorems allow, via the same association that connected the group cohomology with the existence of equivalence principles, the construction of new equivalence principles, at the higher level of (for example) group extensions. These equivalence principles allow the formulation of the laws of nature in a topologically covariant way and the restoration of the fundamental prescription of unitarity required by quantum mechanics (albeit in a modified form) even in the case when topology changing events may occur (as is the case for the formation of black holes). In this way, the observation that the existence of a simple covariant formulation of a theory depends on topological properties of the groups associated to the transformations considered, will become relevant not only for the Galilei and Poincare groups but also for more general situations when a change in topology occurs. Hence, this article aims towards an extension of the equivalence principles as formulated by Einstein in a form that suits better the prescriptions of quantum mechanics.

INDEPENDENCE OF TOPOLOGY AND THE UNIVERSAL COEFFICIENT THEOREM

As argued in the previous chapter, the laws of physics should not depend on unobservable properties of space-time. Specifically the choice of a particular coordinate system or a particular coefficient group in cohomology should not be relevant for the formulation of the laws of physics. I showed in a previous article [9] that specific choices of coefficient groups in cohomology may affect the observable connectedness of space-time (or generally of an abstract space or group) as measured by topological techniques. Here I focus on a different aspect, namely what changes should be made in a theory in order for it to describe the physical reality independent on the way one choses to regard the topology? As has been shown in [4] and as I argued in the previous section, the existence of a trivial second group cohomology associated to a symmetry group implies the existence of a straightforward covariant formulation of the associated theory. The triviality of the cohomology in a given

dimension however, is controlled by the choice of a coefficient structure in the cohomology. The effect of this choice is on its turn, encoded in the UCT. This observation is general and doesn't relate only to the Poincare group. In fact, one can bring the same arguments in the case of the Weyl-Heisenberg group. This encodes the quantization prescriptions and allows a central extension structure. Moreover, its group cohomological properties when analyzed from the perspective of particular coefficient groups allow for the covariant formulation required by the quantum prescription of unitarity. Indeed, this prescription is not preserved in the same form when one changes the coefficient structure used to probe the group topological properties. This is the reason for the paradoxes one encounters when discussing the unitarity in processes involving black holes. I will continue here with a presentation of the topological properties of the Weyl-Heisenberg group, followed by an analogy between the general (or special) relativity covariant formulation and unitarity prescriptions in quantum mechanics. The conclusion of this work shows how to use specific formulations of the universal coefficient theorem in order to restore unitarity when dealing with black holes and event horizons. I will also show how a thermal density matrix appears to be modified when a different choice of a coefficient structure in the (group)-cohomology is made. The final result shows that the notion of density matrix has to be extended such that it incorporates relevant group topological information. Also, entanglement can be connected to the existence of non-trivial group-cohomological classes. Hence, the universal coefficient theorem can show how entanglement is relativized when different coefficient structures are being chosen. This will make subsystems that look completely uncorrelated when analyzed with one coefficient structure, appear entangled when analyzed with another coefficient structure. The information however will always be there, in one situation, encoded in the group law of the actual cohomology and in the other situation in the special form of the extension or torison that appears in the UCT. It has been brought as an argument for the information paradox that a relatively ordered initial situation (dust or a star) leading to a black hole has as an inescapable final state the thermal radiation. Unless some "emission of negative entropy" [1] by the black hole occurs, information should be lost. However, I showed in ref. [9] that the definition of entropy in a situation where several different coefficient groups are required, must change. In fact, the entropy will have to include topological information as well. It will not be defined uniquely. Instead it will have different forms when regarded via different coefficient groups. This allows the changes in entropy required to restore unitarity in a global (topological) way. In order to start this project, I remind the reader that projective representations in physics are required since standard quantum mechanics represents pure states as rays. Because of this,

symmetry operators are represented as classes of unitary ray operators \bar{U} . Unless otherwise stated, in what follows G and K represent general groups. The operation over these classes is defined as

$$\bar{U}(g')\bar{U}(g) = \bar{U}(g'g), \quad g', g \in G \quad (4)$$

Actual operators in each class differ by a phase. Let's make a choice of operators in each of the classes. Let $g, g' \in \Pi$ where Π is a neighborhood of the identity $e \in G$. Now select a representative $U(g'g)$ in the class $\bar{U}(g'g)$. The composition rule becomes then

$$U(g')U(g) = \omega(g', g)U(g'g), \quad |\omega(g', g)| = 1 \quad (5)$$

$\omega(g', g)$ are the local factors that can be written in terms of local exponents as

$$\omega(g', g) = \exp(i\xi(g', g)) \quad (6)$$

Different representatives from each class U' will select new local factors $\omega'(g', g)$. When $U'(g)$ and $U(g)$ belong to the same class they will be related by a phase for each g

$$U'(g) = \gamma(g)U(g), \quad |\gamma(g)| = 1 \quad (7)$$

and this generates a relation between the local factors

$$\omega'(g', g) = \omega(g', g)\gamma^{-1}(g'g)\gamma(g')\gamma(g) \quad (8)$$

If it is possible to select $\gamma(g)$ such that the factors become the identity one says that the local exponents are equivalent to 1. It is however not always possible to extend the choice of representatives around the identity to the whole group. When this can be done the ray representation can be replaced with an ordinary (vector) representation. In general the local factors ω can be seen as mappings

$$\omega : G \times G \rightarrow U(1) \quad (9)$$

satisfying the normalization condition $\omega(e, e) = 1$ and the two-cocycle condition

$$\omega(g'', g')\omega(g''g', g) = \omega(g'', g'g)\omega(g', g) \quad (10)$$

which is nothing but the associativity property of the factors. Two cocycles ω and ω' are equivalent when there exists a two-coboundary

$$\omega_{cob}(g', g) = \gamma^{-1}(g'g)\gamma(g')\gamma(g) \quad (11)$$

such that the two-cocycles are related by

$$\omega'(g', g) = \omega(g', g)\omega_{cob}(g', g) \quad (12)$$

The classes of inequivalent two-cocycles define the second cohomology group $H^2(G, U(1))$. It is important to notice that due to the identification of pure states with classes in the second cohomology group, the fact that states are

pure is dependent on the choice of the coefficients used to probe the desired space, hence dependent on the coefficient group in cohomology. This has a major impact on the identification of the thermal final state in the case of a black hole. The "appearance" of the radiation as thermal (or the states as mixed) depends on a specific choice of coefficient groups. A topologically covariant formulation however can show that the "locally-thermal" radiation will in fact contain global, topological information. The operators inside a class $\tilde{U}(g)$ can be written as $e^{i\theta}U(g)$. In this way I introduced a new variable θ . In this case the transformation rule becomes

$$e^{i\theta'}U(g')e^{i\theta}U(g) = e^{i(\theta'+\theta)}e^{i\xi(g',g)}U(g'g) = e^{i\theta''}U(g'') \quad (13)$$

One can use the notation ($\zeta = e^{i\theta}$, $\omega(g',g) = \exp(i\xi(g',g))$) and form a new group \tilde{G} with the parameters (ζ, g) such that \tilde{G} contains $U(1)$ as an invariant subgroup and $\tilde{G}/U(1) = G$ i.e. \tilde{G} is a (central) extension of G by $U(1)$. Following the rationale of this article, the next step is to formulate a quantum analogue. For this we construct the Weyl-Heisenberg group as a manifold (q, p, ζ) with the composition law given by

$$\begin{aligned} q'' &= q' + q \\ p'' &= p' + p \\ \zeta'' &= \zeta' \zeta \exp\left(\frac{i}{2\hbar}(q'p - p'q)\right) \\ (\zeta^{-1}; q, p)^{-1} &= (\zeta^{-1}; -q, -p) \end{aligned} \quad (14)$$

Here, the two-cocycle is given by

$$\xi(g', g) = \frac{1}{2\hbar}(q'p - p'q) \quad (15)$$

Again, this group can be seen as a $U(1)$ extension of the $2n$ dimensional abelian (p, q) group. The standard quantum construction in terms of the Dirac bra-ket formalism relies on the possibility of formulating the quantization prescription in a covariant form. This depends on the second group cohomology of the associated symmetry transformation.

More practically let for example (C_*, ∂) be a chain complex over a ring R and let M be the associated module. The chain groups are C_* . Then there is a map

$$\text{Hom}_R(C_q, M) \times C_q \rightarrow M \quad (16)$$

that evaluates like

$$(f, z) \rightarrow f(z) \quad (17)$$

This is a general formulation of a structure that has analogues in the covariant and contravariant objects in general relativity but also in the bra-ket notation of standard

quantum mechanics. In quantum mechanics the amplitudes are characterized by complex numbers. The adjoint is defined naturally via hermitian conjugation giving rise to the bra-ket formalism and allowing the construction of theories preserving overall unitarity. In general relativity adjoints are constructed as dual 1-forms that appear as "covariant" indices and together with their contravariant counterparts assure that the theory can be formulated in a diffeomorphism invariant form despite the possible intrinsic curvature of spacetime. In principle the 1-forms take the value of a vector and produce a scalar. If \tilde{P} is a 1-form and \vec{V} is a vector then $\langle \tilde{P}, \vec{V} \rangle = \tilde{P}(\vec{V}) = \vec{V}(\tilde{P})$. The existence of such a covariant formulation and the associated equivalence principle is related to the triviality of the second cohomology group associated to the considered symmetry of the theory. This symmetry might be described by a (possibly central) extension of the original group. Up to now, the statements regarding equivalence principles have been constructed only at the level of symmetries generated by operators forming groups or semigroups. In a more physical language, the statements of Galilei and Einstein, namely that the laws of nature should be written in a form that remains unchanged to a change of coordinates imply the construction of covariant formulations in terms of vectors, tensors, spinors, etc. In the context of the Galilei group the existence of a covariant formulation is obstructed by the fact that its second group cohomology is non-trivial. This is due to the fact that the time component is absolute. Going to a relative time alters the group structure in a way that makes a covariant formulation manifest and trivializes the group cohomology, leading to the Poincare group. However, there are physical and logical indications that the laws of nature should also be written in a form that is independent on arbitrary choices of coefficient groups in (co)homologies. This statement implies that the laws of nature should not depend on a particular choice of probing the topological properties of a space or a group. However, in order to construct a theory of this form, it appears to be necessary to go beyond symmetry groups of a given, fixed cohomological structure when formulating the equivalence principles. One method to do so is given by the universal coefficient theorems. These theorems state that a specific framework, constructed by the choice of a coefficient group in (co)homology is (up to (extension) torsion in (co)homology) equivalent with the choice of an integer coefficient group. One result of this theorem is that distinct classes in (co)homology under one coefficient group may appear as identified under another coefficient group. Suppose M is a module over R then the sequence:

$$0 \rightarrow Ext_R(H_{q-1}(C_*), M) \rightarrow H^q(C_*; M) \rightarrow Hom(H_q(C_*), M) \rightarrow 0 \quad (18)$$

is exact. Here Ext is the group extension. It appears then that cohomology groups that look non-trivial given a coefficient structure become trivial under another one. There are several ways in which possible pairings as the ones discussed above can be mapped into the realm of universal coefficient theorems. One possible pairing defined in the way described above is

$$\langle, \rangle: H^q(C_*; M) \times H_q(C_*) \rightarrow M \quad (19)$$

which relates homology with cohomology. This pairing is bilinear and its adjoint is a homomorphism

$$H^q(C_*, M) \rightarrow Hom(H_q(C_*); M) \quad (20)$$

Universal coefficient theorems, among other things, provide a measure of how this adjoint fails to be an isomorphism in terms of Ext^q and Tor_q [10]. Here q represents the dimension of the space for which the (co)homology is calculated. Particularizing this statement in the previous cases, one may find that the bra-ket formulation of standard quantum mechanics as well as the covariant formulation of general relativity must be adapted in the situation when the measurement dependence of the topology of the space(time) becomes relevant i.e. when the effects depending on the way in which the coefficient structure is chosen in the cohomology are important.

BLACK HOLES AND THE UNITARITY PROBLEM

The previous sections showed that when using equivalence principles and covariant formulations of theories, one usually relies on specific topological properties of the symmetry groups. Especially the second group-cohomology, when trivial, allows for a simple covariant formulation as the one used in the bra-ket formalism or in the tensorial construction of general relativity. However, not in all situations is the second group-cohomology trivial. The nature of the second cohomology depends on one side on the manifold acted upon by the group and on the other side on the coefficient structure chosen in order to describe the cohomology itself. When the second cohomology of the required group is non-trivial one can still formulate a covariant theory provided one considers the universal coefficient theorem and the specific extensions and/or torsions. In this section, I present some physical arguments for the necessity of a coefficient independent construction and, implicitly, of theories that do not depend on arbitrary changes of topology. If in the case of general relativity and quantum mechanics the

covariance had to be implemented with respect to a symmetry group, in order to implement the topological covariance one has to consider the coefficient structures in (co)homology. The scalars, vectors and tensors of general relativity will have their equivalents in the various morphisms between extensions or torsions in the universal coefficient theorems. Probably the most important object for which the current discussion is relevant is a black hole. The problem of information conservation was discussed in the context of quantized fields over a given background in [1]. I partially follow the discussion presented therein, pinpointing the aspects where an extension of that treatment is necessary due to some ignored topological aspects. Considering, in agreement with [1] a massless Hermitian scalar field and an uncharged non-rotating black hole, after quantization one obtains a scalar field operator ϕ which satisfies the wave equation

$$\square\phi = 0 \quad (21)$$

Given the background metric associated to the Schwarzschild spacetime [5] where the considered black hole is present one can rewrite this as

$$(-g)^{\frac{1}{2}}\partial_\mu[(-g)^{\frac{1}{2}}g^{\mu\nu}\partial_\nu\phi] = 0 \quad (22)$$

One can also define a conserved scalar product of the form

$$(\phi_1, \phi_2) = i \int d^{n-1}x |g|^{1/2} g^{0\nu} \phi_1^*(x, t) \overleftrightarrow{\partial}_\nu \phi_2(x, t) \quad (23)$$

the integral being over a constant t hypersurface. When ϕ_1 and ϕ_2 are solutions of the field equation above and vanish at spatial infinity, then (ϕ_1, ϕ_2) is conserved. The existence of a flow of particles originating at a small affine distance from the event horizon has been derived in [2]. One particularity of this derivation is that the average number of outgoing particles in each mode is distributed in accordance with a thermal spectrum. Moreover, the full probability distribution, not just the average, of the emitted particles is that of thermal radiation. This observation creates a conflict with standard quantum mechanics when one considers the process of an in-falling object together with the radiation emitted on the external part of the horizon. The main issue is that this process does not preserve unitarity. If the in-falling system is in a pure quantum state, the out-coming radiation is in a naturally mixed state. The full information related to the in-falling object is forever hidden behind the horizon. This result, however, appears only when one does not consider the process as described in a topologically covariant way. Using some of the observations in [9] I show

here that there exists a special choice of coefficients in the field space cohomology for which there exists a unitary connection between the supposed thermal radiation and the in-falling system. This suggests that the quantum information is in fact conserved, albeit not in the obvious way, directly in the fields, but in the topology (more precisely in the higher cohomology) of the automorphisms of the field space. In order to show this I continue the derivation of the spectrum of the Hawking radiation underlining the modifications in the way of thinking that must be considered in order to obtain the correct result. This method is in agreement with the AdS/CFT solution but its construction allows for a higher degree of generality. Let me now take the quantum fields used in the field equation above and decompose them as

$$\phi = \int d\omega (a_\omega f_\omega + a_\omega^+ f_\omega^*) \quad (24)$$

where f_ω and f_ω^* form a complete set of solutions of the field equation and are normalized according to

$$(f_{\omega_1}, f_{\omega_2}) = \delta(\omega_1 - \omega_2) \quad (25)$$

The a_ω operators are time independent. The standard method of quantization would be

$$\begin{aligned} [a_{\omega_1}, a_{\omega_2}^+] &= \delta(\omega_1 - \omega_2) \\ 0 &= [a_{\omega_1}^+, a_{\omega_2}^+] = [a_{\omega_1}, a_{\omega_2}] \end{aligned} \quad (26)$$

Let me chose the f_ω such that at early times and large distances they form a complete set for the incoming positive frequency solutions of energy ω . It is possible to compute the spectrum of the created particles by making an expansion of the field in terms of the late time positive frequency solutions. Let p_ω be the solutions of the field equation that have zero Cauchy data on the event horizon and are asymptotically out-coming with positive frequency. Again, consider that in this domain p_ω and p_ω^* form a complete set of solutions. The normalization condition is

$$(p_{\omega_1}, p_{\omega_2}) = \delta(\omega_1 - \omega_2) \quad (27)$$

There must also be an in-coming component of the solution at the event horizon at late times. Let me call this set of solutions q_ω . The superposition of these components at late times is localized on the horizon and has zero Cauchy data on the distant region. The components q_ω and q_ω^* form a complete set on the horizon and are normalized as

$$(q_{\omega_1}, q_{\omega_2}) = \delta(\omega_1 - \omega_2) \quad (28)$$

The two components, being defined in disjoint regions are assumed to have null scalar product

$$(q_{\omega_1}, p_{\omega_2}) = 0 \quad (29)$$

The expansion of the fields in terms of the above components is then

$$\phi = \int d\omega \{b_\omega p_\omega + c_\omega q_\omega + b_\omega^+ p_\omega^* + c_\omega^+ q_\omega^*\} \quad (30)$$

where b_ω and c_ω are the associated annihilation operators. The commutation relations are now

$$\begin{aligned} [b_{\omega_1}, b_{\omega_2}^+] &= \delta(\omega_1 - \omega_2) \\ [c_{\omega_1}, c_{\omega_2}^+] &= \delta(\omega_1 - \omega_2) \end{aligned} \quad (31)$$

all other commutators are vanishing. The spectrum of the outgoing particles is determined by the coefficients of the Bogolubov transformation relating b_ω to $a_{\omega'}$ and $a_{\omega'}^+$. One may define the operators c_ω and c_ω^+ as the annihilation and creation operators for particles falling into the black hole. However, this definition is ambiguous due to the fact that positive frequency components for the in-falling matter is not well defined. The physical meaning of these operators should therefore be takes as symbolic. Using the complete set given by f_ω and f_ω^* one can write

$$p_\omega = \int d\omega' (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f_{\omega'}^*) \quad (32)$$

where α and β are complex numbers independent of the coordinates. We can therefore calculate

$$b_\omega = (p_\omega, \phi) \quad (33)$$

and expressing ϕ and p_ω in terms of $f_{\omega'}$ and $f_{\omega'}^*$, one can obtain

$$b_\omega = \int d\omega' (\alpha_{\omega\omega'}^* a_{\omega'} - \beta_{\omega\omega'}^* a_{\omega'}^+) \quad (34)$$

and the invariant becomes

$$(p_{\omega_1}, p_{\omega_2}) = \int d\omega' (\alpha_{\omega_1\omega'}^* \alpha_{\omega_2\omega'} - \beta_{\omega_1\omega'}^* \beta_{\omega_2\omega'}) \quad (35)$$

It is worthwhile noticing that the coefficients can be expressed as

$$\begin{aligned} \beta_{\omega\omega'} &= -(f_{\omega'}^*, p_\omega) \\ \alpha_{\omega\omega'} &= (f_{\omega'}, p_\omega) \end{aligned} \quad (36)$$

The discussion up to this point is unsurprising. The calculation of the coefficients above can be used in order to derive the average number of created particles observed at later times. However the exact form in which the previous calculations are being performed does not take the fact into account that the topology as encoded by cohomology groups changes when a black hole forms. In order to show this one has to recall the abstract formulation of the bracket notation used in the previous chapter. While the curvature of spacetime is correctly taken into

account in the previous discussion, there are certain modifications required for the pairing operations used above to be isomorphically translated from the language of flat or curved spacetime to the language of spacetime with a horizon. I remind the reader here the mathematical definition of a topology:

Let X be a non-empty space. A collection τ of subsets of X is said to be a topology on X if

- X and the empty set belong to τ
- the union of any (finite or infinite) number of sets in τ belongs to τ
- the intersection of any two sets in τ belongs to τ

In this case the pair (X, τ) is called a topological space. It is important to notice that there are several possible choices of topologies over a space. One possible choice would be to consider any two points joined together in a subset for a specific topology if they can be connected by light in both directions. The space made up of low density dust before the formation of a black hole has every point connected in such a topology. Once a horizon forms the topology defined in the above way changes. Moreover, after the horizon is formed, any topology that, prior to the formation of the horizon, connected two points on different sides of what is now the horizon, must change in order to consider the new situation.

Because of this, each of the constructions defined above has to be redefined. Consider first the conserved scalar product over the field space:

$$(\phi_1, \phi_2) = i \int d^{n-1}x |g|^{1/2} g^{0\nu} \phi_1^*(x, t) \overleftrightarrow{\partial}_\nu \phi_2(x, t) \quad (37)$$

Such a scalar product depends on the topology of the field space at least at the level of the second cohomology. The two fields appearing in the above inner product are solutions of the wave equation. Their form is correct and needs no modification when a topology change occurs. However, the general pairing of such fields in the context of a non-trivial topology must take into account the universal coefficient theorem. In this way, a sampling of the field space is needed. This sampling requires a form of triangulation. In order to show the basic ideas one needs several definitions. First one has to define a (geometric) q -simplex Δ^q as

$$\Delta^q = \{(t_0, t_1, \dots, t_q) \in \mathbb{R}^{q+1} \mid \sum t_i = 1, t_i \leq 0 \forall i\} \quad (38)$$

Here, q represents the dimension of the simplex. The face maps are functions relating consecutive dimensions of the simplex

$$f_m^q : \Delta^{q-1} \rightarrow \Delta^q \quad (39)$$

defined by adding an extra coordinate from the origin for the higher dimension

$$(t_0, t_1, \dots, t_{q-1}) \rightarrow (t_0, \dots, t_{m-1}, 0, t_m, \dots, t_{q-1}) \quad (40)$$

In order to represent a space, this abstract construction must be mapped into a space X . In order to do this a continuous map is required

$$\sigma : \Delta^q \rightarrow X \quad (41)$$

Then, any space can be constructed as a chain

$$\{X\} = \sum_{i=1}^l r_i \sigma_i \quad (42)$$

where $\{r_i\}$ is the set of coefficients belonging in general to a ring R . The space X as seen via the basis formed from the q -simplices defined above is denoted $S_q(X; R)$. The boundary map is defined as

$$\partial : S_q(X; R) \rightarrow S_{q-1}(X; R) \quad (43)$$

such that

$$\partial(\sigma) = \sum_{m=0}^q (-1)^m \sigma \circ f_m^q \quad (44)$$

One can extend the above definition by introducing the covariant functor $S_*(-; R)$. This means that given a continuous map

$$f : X \rightarrow Y \quad (45)$$

this will induce a homomorphism

$$f_* : S_*(X; R) \rightarrow S_*(Y; R) \quad (46)$$

with the definition

$$f_*(\sigma) = f \circ \sigma \quad (47)$$

Then, the complex $(S_*(X; R), \partial)$ is called the simplicial chain complex of the space X with coefficients in R . The homology of this chain complex with coefficients in R is then

$$H_q(X; R) = \frac{\ker \partial}{\text{Im } \partial} \quad (48)$$

where \ker represents the kernel of the considered map and Im represents its image. The formal inversion of the arrow in the boundary operator generates in the same way the cohomology of the chain complex. One may ask what happens if a structure of this form is used in order to map a space before and after the collapse of a dust cloud into a black hole. While all the simplexes can be defined in the initial case, after the formation of a horizon some subtleties arise. If the definition of the topology is such that points separated by a horizon are not defined to belong in the same open set then the simplex structure above must be altered. However, there is no physical difficulty in extending the metric of spacetime beyond the horizon. Also, particles can fall through the horizon.

In order to maintain a topological covariant description, the change must therefore be made via the coefficients of the simplexes as defined above. Because of this, several concepts required in the construction of the Hawking radiation and the derivation of its distribution function will have to be adapted. First, any pairing that is required for the definition of an invariant structure must be constructed via the universal coefficient theorem. It is the extension that controls the pairing and the UCT provides the information about what is "lost" when one makes a change in the topology via the coefficient structure. This will have some effect on the definition of entanglement, the construction of density matrices, etc. Second, I will show that the correction given by the "lost information" encoded in the extension group appears in the form of an extra factor in the composition rule. In order to do this I will refer again to [4]. One object I will require in the next part is the exact sequence. Let $f_i : G_i \rightarrow G_{i+1}$ be a collection of group homomorphisms, then the sequence

$$\dots \rightarrow G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} G_{i+2} \rightarrow \dots \quad (49)$$

is called exact if

$$\text{Im } f_i = \ker f_{i+1} \quad (50)$$

As a result, for any exact sequence $f_i \circ f_{i-1} = 0$. Using this formulation, let G and K be two abstract groups. A group \tilde{G} is said to be an extension of G by K if K is an invariant subgroup of \tilde{G} and $\tilde{G}/K = G$. In terms of exact sequences this means that

$$1 \rightarrow K \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \quad (51)$$

is exact i.e. K is injected into \tilde{G} and \tilde{G} is projected onto G by the canonical homomorphism so that $G = \tilde{G}/K$. However, the mere knowledge of K and G does not define \tilde{G} uniquely. In order to be able to discern extensions one has to define two exact sequences

$$1 \rightarrow K \xrightarrow{i_1} \tilde{G}_1 \xrightarrow{\pi_1} G \rightarrow 1 \quad (52)$$

$$1 \rightarrow K \xrightarrow{i_2} \tilde{G}_2 \xrightarrow{\pi_2} G \rightarrow 1 \quad (53)$$

If the two group extensions are related via an isomorphism \tilde{f} :

$$\tilde{f} : \tilde{G}_1 \rightarrow \tilde{G}_2 \quad (54)$$

and the injective maps $i_{1,2}$ and the projections $\pi_{1,2}$ satisfy

$$\begin{aligned} i_2 &= \tilde{f} \circ i_1 \\ \pi_1 &= \pi_2 \circ \tilde{f} \end{aligned} \quad (55)$$

then the extensions are equivalent. Consider now the two group extensions, defined by two different two-cocycles ξ_1 and ξ_2 with their group laws defined separately with simple brackets (...) for the first group and square brackets [...] for the second group:

$$(\theta', g')(\theta, g) = (\theta' + \theta + \xi_1(g', g), g'g), \quad [\theta', g'][\theta, g] = [\theta' + \theta + \xi_2(g', g), g'g] \quad (56)$$

If there exists an isomorphism \tilde{f} as defined above and if we can rewrite

$$(\theta, g) = (\theta, e)(0, g) \quad (57)$$

$(0, e)$ being the identity of this law, \tilde{f} is completely determined when the images of (θ, e) and $(0, g)$ are given. From the conditions on the injection and projection above one obtains

$$\begin{aligned} \tilde{f} \circ i_1 = i_2 &\Rightarrow \tilde{f}(\theta, e) = [\theta, e] \\ \pi_2 \circ \tilde{f} = \pi_1 &\Rightarrow \tilde{f}(0, g) = [\eta(g), g] \end{aligned} \quad (58)$$

This implies a general form for \tilde{f} namely

$$\tilde{f}(\theta, g) = [\theta + \eta(g), g] \quad (59)$$

The knowledge of η determines the knowledge of \tilde{f} . However, \tilde{f} is also a homomorphism hence

$$\tilde{f}(\theta' + \theta + \xi_1(g', g), g'g) = [\theta' + \theta + \xi_1(g', g) + \eta(g'g), g'g] \quad (60)$$

must be equal to

$$\begin{aligned} \tilde{f}(\theta', g')\tilde{f}(\theta, g) &= [\theta' + \eta(g'), g'][\theta + \eta(g), g] = \\ &= [\theta' + \theta + \xi_2(g', g) + \eta(g') + \eta(g), g'g] \end{aligned} \quad (61)$$

and hence

$$\begin{aligned} \xi_1(g', g) &= \xi_2(g', g) + \eta(g') + \eta(g) - \eta(g'g) = \\ &= \xi_2(g', g) + \xi_{cob}(g', g) \end{aligned} \quad (62)$$

where the notation $\xi_{cob}(g', g)$ is used for the two-coboundary generated by $\eta(g)$. The calculation above gives a condition for the equivalence of extensions. One can see that proportional two-cocycles $\xi_2 = \lambda \xi_1$ may define equivalent groups but inequivalent extensions. In order to make the connection with the bracket construction and to classify the extensions one has to rely on a fiber bundle definition of the extension. Let therefore G and K be abstract general groups and \tilde{G} be the extension of G by K . One can relate the cosets of K in \tilde{G} , each defining an element $g \in G$ with the fibers over g of a fiber bundle that defines the extension. The fiber through $\tilde{g}_0 \in \tilde{G}$ is given by

$$\pi^{-1}(\pi(\tilde{g}_0)) = \{\tilde{g}|\tilde{g} = k\tilde{g}_0, k \in K\} \quad (63)$$

A section of $\tilde{G}(K, \tilde{G}/K = G)$

$$s : G \rightarrow \tilde{G}, \quad s : (g) \rightarrow s(g) \quad (64)$$

selects an element in \tilde{G} in each fiber. Now, given a fiber

$$\pi(s(g'')) = \pi(s(g')s(g)) \quad (65)$$

thus there exists a factor $\omega(g', g) \in K$ such that

$$s(g')s(g) = \omega(g', g)s(g', g) \quad (66)$$

and this relation defines the factor $\omega(g', g)$. One can define $\omega(g', e) = \omega(e, g) = s(e)$ and take $s(e) = \tilde{e} \in \tilde{G}$. Thus, one obtains the normalized section. Similarly one can obtain, for a normalized section, also a normalized factor:

$$\omega(g, e) = \omega(e, g) = \omega(e, e) = e \in K \quad (67)$$

As a general statement, relative to any normalized trivializing section $s : G \rightarrow \tilde{G}$ one can associate a factor system $\omega : G \times G \rightarrow K$ satisfying

$$\omega(g'', g)\omega(g'g', g) = ([s(g'')]\omega(g', g))\omega(g'', g'g) \quad (68)$$

where $[s(g)]k = s(g)ks(g)^{-1} \forall k \in K$. According to this fiber bundle representation of the extensions, the group law of the group extension can be defined in terms of the factor system as

$$(k'', g'') = (k', g') *_s (k, g) = (k'[s(g')]k\omega(g', g), g'g) \quad (69)$$

Returning to the physical problem, the invariant bracket defined above,

$$(\phi_1, \phi_2) = i \int d^{n-1}x |g|^{1/2} g^{0\nu} \phi_1^*(x, t) \overleftrightarrow{\partial}_\nu \phi_2(x, t) \quad (70)$$

must be extended in order to obtain a topologically covariant description. The change in topology can be considered as the effect of a specific choice of the coefficient structure in (co)homology. The definition of the adjoint of the topological bracket can be identified as the right

hand side of the universal coefficient theorem. When a choice of coefficients is considered such that the horizon of the black hole becomes visible one obtains a correction to the bracket as given by the factor that characterizes the extension of the homology group in a dimension smaller by one unit. It will be this extension that will generate the algebra to be used in the physical situation defined by the coefficient structure where the horizon is visible. The bracket is defined now with a correction in the group operation associated to its defining symmetry. Hence a topological factor would be missing in the construction used in [1]. I underline that this factor is purely topological and in a sense of a quantum nature. Hence one has to extend the scalar bracket when a topological covariance is required:

$$(\phi_1, \phi_2) = i \int d^{n-1}x |g|^{1/2} g^{0\nu} \phi_1^*(x, t) \overleftrightarrow{\partial}_\nu \phi_2(x, t) \quad (71)$$

must be transformed into

$$(\phi_1, \phi_2)' = \langle \phi_1, \phi_2 \rangle \omega(\phi_1, \phi_2)(\phi_1, \phi_2) \quad (72)$$

where the $\langle \dots \rangle$ notation refers to the topological invariant and $\omega(\phi_1, \phi_2)$ refers to the factor system that characterizes the extension and depends on the choice of the coefficient structure. This factor will appear also in the coefficients defining the probability of particle detection far from the black hole horizon. The fact that an object can fall behind the horizon while nothing can travel from behind the horizon to the outside will imply the change in the topology used to define the considered phenomenon in the presence of a black hole. This change will be encoded in the factor system. It will however not be visible in any perturbative analysis. To make these considerations more accurate I will follow again [1].

Consider therefore the vacuum state at the infinite past as

$$|0_- \rangle = \sum \sum \lambda_{AB} |A_I \rangle |B_H \rangle \quad (73)$$

where $|A_I \rangle$ is the outgoing state with n_{ja} particles in the j th outgoing mode and $|B_H \rangle$ is the horizon state with n_{kb} particles in the k th mode going into the hole. Otherwise stated

$$\begin{aligned} |A_I \rangle &= \prod_j (n_{ja}!)^{-1/2} (b_j^+)^{n_{ja}} |0_I \rangle \\ |B_H \rangle &= \prod_k (n_{kb}!)^{-1/2} (c_k^+)^{n_{kb}} |0_I \rangle \end{aligned} \quad (74)$$

One can chose an observable at the far future, composed only of $\{b_j\}$ and $\{b_j^+\}$ and operating only on the vectors $|A_I \rangle$. The expectation value of this observable can be written as

$$\langle 0_- | Q | 0_- \rangle = \sum \sum \rho_{AC} Q_{CA} \quad (75)$$

where $Q_{CA} = \langle C_I | Q | A_I \rangle$ is the matrix element of the observable in the Hilbert space of the outgoing states. The density matrix is

$$\rho_{AC} = \sum \lambda_{AB} \bar{\lambda}_{CB} \quad (76)$$

and is associated to measurements in the far future but not to measurements of systems falling into the black hole. It is at this point where several extensions of the standard prescription are necessary. This density matrix does not encode the full information that can be obtained

in the far future. It does encode however everything that can be obtained from non-topological considerations. In order to see this one has to observe the fact that the information can be encoded not only directly, as considered here, but also via the cohomology groups associated to the field space. I showed in [9] that quantum observables are relative, depending on the particular choice of a coefficient group in the cohomology of the field space. A particular form of the universal coefficient theorem is

$$0 \rightarrow Ext_R^1(H_{i-1}(X; R), G) \rightarrow H^i(X; G) \xrightarrow{h} Hom_R(H_i(X; R), G) \rightarrow 0 \quad (77)$$

This can be interpreted in a form that resembles the interpretation of the non-commutativity of some physical observables: the third arrow

$$H^i(X; G) \xrightarrow{h} Hom_R(H_i(X; R), G) \quad (78)$$

maps the cohomology with coefficients in the group G into the homomorphisms between the homology with coefficients in R and the group G . The sequence is exact, hence this map is a surjection. This means there are no elements in the set of homomorphisms from the homology with coefficients in R to the group G not represented in the cohomology with coefficients in G . However, there are elements in the cohomology that can be mapped into the same element of Hom . The second arrow

$$Ext_R^1(H_{i-1}(X; R), G) \rightarrow H^i(X; G) \quad (79)$$

is an injection. Hence the extension encodes the way in which the use of a coefficient structure instead of another

changes the classes of the cohomology.

One can extend the uncertainty principle from the non-commuting observables to the mutually incompatible coefficient structures in cohomology. Indeed, the universal coefficient theorem shows that physical observables in a quantum field theory on a topological space are relative, depending on a particular choice of the coefficient group in the cohomology. Observables visible when using one coefficient structure for the probing of the functional space may become indistinguishable when another coefficient structure, incompatible with the first, is used. This fact can be translated in terms of density matrices. Indeed one can construct a density matrix in the form given above

$$\rho = \sum_i \rho_i |\Psi_i\rangle\langle\Psi_i| \quad (80)$$

which can be represented in an arbitrary basis as

$$\rho = \sum_{i,a,b} \rho_i |\phi_a\rangle\langle\phi_a| \Psi_i \langle\Psi_i| \phi_a\rangle\langle\phi_a| = \sum_{ab} |\phi_a\rangle\langle\phi_b| \rho_{ab} \quad (81)$$

The expectation value of an observable can be defined as

$$\bar{F} = tr[\rho F] = \sum_{ab} F_{ba} = \sum_{ab} \rho_{ab} \langle\phi_b| F |\phi_a\rangle \quad (82)$$

I showed in [9] (Theorem 2) that the discernibility of quantum states is relative in the sense that it depends on the choice of a coefficient group in the cohomology. Here, I show a consequence of this. Indeed, let now take a system composed of two subsystems identified by the

variables q_1 and q_2 . Suppose the entire system is in a pure state and let that state be $|\Psi_{12}\rangle$. If this state can be factorized into a product of pure states from subsystem 1 and subsystem 2 as

$$|\Psi_{12}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \quad (83)$$

then the subsystems are said to be unentangled. Otherwise the systems are said to be entangled. However, this notion cannot be described unambiguously in the pres-

ence of horizons because there exists at least one choice of coefficients in cohomology where the subsystems are entangled and one choice where the subsystems are independent while both choices being compatible with the region inside and outside the horizon. It is always possible to traverse the horizon towards the interior of the black hole, hence the physics should not change due to a choice of topology or a choice of a coefficient group.

The condition for this is translated in the isomorphism condition for the extensions, formulated in the previous section. In terms of cocycles this leads to the fact that the density matrices must differ in an additive coboundary. One should not have a difference between the two density matrices as seen via one coefficient group and the other of the form $\xi_1 = \lambda \xi_2$ as this relation cannot insure the isomorphism of the extensions. Hence, the density matrix must be extended additively, leading to terms that break the factorization into pure states. Otherwise stated, pure states can be seen as classes in the second cohomology group $H^2(G, U(1))$ associated to the above mentioned group. The universal coefficient theorem implies that the classes can merge or dissociate according to the coefficient groups used to map the analyzed space (or group). Hence, as the notion of "entangled" or "unentangled" is well defined in flat or curved space, it becomes a relative notion when the necessity of a topologically covariant description arises. Another way of looking at this is to see that the non-trivial commutation relations appear as two-cocycles in the cohomology associated to the Weyl-Heisenberg group in particular, as shown in the previous section, and in general, the non-commuting property of two general observables which leads to the block structure of the density matrix depends on the choice of coefficient groups in the associated cohomology. Hence, the "uncertainty principle" introduced at the level of topological information via the universal coefficient theorem can be mapped directly into block diagonal elements of the density matrix. Hence, quantum correlation arises as a global topological property when a horizon that enforces two different choices of coefficient groups appears.

Of course, this observation may have implications not only for black holes but also for entangled states in topological condensed matter systems. This may however be the subject of a future work.

One may ask if locality is preserved in this situation. Indeed, the problem of locality when unitarity is restored appears to be fundamental to the AdS/CFT solution of the information paradox [6]. The information, in the approach of this work, is encoded in the global topological structure of the field in such a way that it is not accessible by any local measurements. One has to remember that the quantum field is not a measurable quantity. There is no physically observable "quantum field" in the same way in which there is no physically observable wavefunction. Nevertheless, the global, topological properties of the fields (and wavefunctions) are important and encode

relevant information. Any local measurement can be seen as a "small" (weak) measurement. Can such a measurement reveal the global information? The correct answer to this question is no. Any weak measurement will reveal a weak information that will not provide any access to the information encoded globally and retrievable only via a statistical topological measurement. If one chooses a coefficient structure for which the global non-triviality is invisible, locality is regained. Information is conserved but only in the factors appearing due to the use of the extension group. Hence unitarity is still preserved but in a "hidden" form (in the extension). If one chooses a suitable coefficient structure the global information becomes accessible due to the manifest visibility of the global non-triviality. However, one cannot recover the information unless one performs a probing of the topology. This may look non-local in a sense but the information obtained in this way concerns topologically non-trivial field (wavefunction) structures hence this "non-locality" is not a physical one but rather one related to a choice of performing certain measurements.

CONCLUSION

As a conclusion, I have shown that topological corrections to the thermal radiation of a black hole as given by the requirement of topological covariance of the laws of physics can account for a factor in the coefficients defining the thermal radiation. This factor imposes non-trivial changes in the form of the distribution function that amount to non-thermal corrections. This observation confirms the suspicions that the solution of the unitarity problem relies on non-perturbative effects and on topological properties of the quantum groups involved in the derivation of the radiation distribution function.

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- [1] S. Hawking, Phys. Rev. D 10 (14), 2460 (1976)
 - [2] S. Hawking, Commun. Math. Phys. 43, 199 (1975)
 - [3] A. Einstein, Ann. d. Physik 354 (7), 769-822 (1916)
 - [4] J. A. Azcarraga, M. Josi, Lie groups, Lie algebras, cohomologies and some applications in physics, See page 291 for the connection between the topological structure of the Galilei and Poincare groups and the existence of a simple covariant formulation.
 - [5] Schwarzschild, K. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften 7: 189196 (1916)
 - [6] D. A. Lowe, J. Polchinsky, L. Thorlacius, J. Uglum, Phys. Rev. D 52 6997 (1995)
 - [7] N. Bogoliubov, J. Phys. (USSR), 11, p. 23 (1947)
 - [8] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231
 - [9] A. T. Patrascu, Phys. Rev. D 90, 045018 (2014)
 - [10] J. F. Davis, P. Kirk, Lecture Notes in Algebraic Topology (see page 43 and 47 in notes)